Topic: Sequence of transformations

Question: Which of the following is a sequence of transformations that can be applied to yield the graph of the function?

$$6\left(\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)-5\right) \text{ from } \sin\theta$$

Answer choices:

A A shift to the right, a horizontal compression, a vertical stretch, and a downward shift.

B A shift to the left, a horizontal stretch, a vertical compression, and an upward shift.

C A shift to the left, a horizontal compression, a vertical compression, and a downward shift.

D A shift to the right, a horizontal stretch, a vertical stretch, and a downward shift.

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Solution: D

We'll begin by rewriting the expression for the function

$$6\left(\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)-5\right)$$

in the form $a\sin(b(\theta+c))+d$, where a, b, c and d are real numbers and b is positive.

Notice that the 6 is to be distributed over the sine term and the -5, so we'll do that first:

$$6\left(\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)-5\right)=6\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)+6(-5)$$

$$6\left(\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)-5\right)=6\sin\left(\frac{2\theta}{5}-\frac{\pi}{10}\right)-30$$

Next, we'll express

$$\frac{2\theta}{5}-\frac{\pi}{10}$$

in a way that will enable us to determine the values of b and c.

Factoring $2/5$ out of each of the two terms, we obtain

$$\frac{2\theta}{5}-\frac{\pi}{10}=\left(\frac{2}{5}\right)\left[\left(\frac{5}{2}\right)\left(\frac{2\theta}{5}\right)-\left(\frac{5}{2}\right)\left(\frac{\pi}{10}\right)\right]$$

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$$\frac{2\theta}{5} - \frac{\pi}{10} = \left(\frac{2}{5}\right)\left(\theta - \frac{\pi}{4}\right)$$

Combining these results, we find that the expression for the function

$$6\left(\sin\left(\frac{2\theta}{5} - \frac{\pi}{10}\right) - 5\right)$$

can be rewritten as

$$6\sin\left(\left(\frac{2}{5}\right)\left(\theta - \frac{\pi}{4}\right)\right) - 30$$

This tells us that $a = 6$, $b = 2/5$, $c = -\pi/4$, and $d = -30$.

Since $c = -\pi/4$, which is negative, the first transformation is a horizontal shift of $-\pi/4$ (i.e., a shift of $\pi/4$ units to the right).

Since $b = 2/5$, which is less than 1, the second transformation is a horizontal stretch of $2/5$. That is, it increases the period from 2π to

$$\frac{2\pi}{\left(\frac{2}{5}\right)}\left(= 2\pi\left(\frac{5}{2}\right) = 5\pi\right)$$

Since $a = 6$, which is greater than 1, the third transformation is a vertical stretch of 6. That is, it increases the amplitude from 1 to $|6|\ (= 6)$.

And since $d = -30$, which is negative, the fourth (and last) transformation is a vertical shift of -30 (i.e., a downward shift of 30 units).

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Question: Which of the following functions has a graph that can be constructed from the graph of cosine by only a vertical shift and a horizontal compression?

Answer choices:

A $\cos\theta - 2.9$

B $2\cos\left(\theta + \frac{\pi}{8}\right)$

C $\cos(1.5\,\theta) + 0.2$

D $\cos\left(0.25\,\theta - \frac{\pi}{3}\right)$

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Solution: C

The function given in answer choice C,

$$\cos(1.5\,\theta) + 0.2$$

can be written as

$$\cos\left(\frac{3\theta}{2}\right) + 0.2 \left(= \cos\left(\left(\frac{3}{2}\right)(\theta + 0)\right) + 0.2\right)$$

This is in the form $a\cos(b(\theta + c)) + d$, with $a = 1$, $b = 3/2$, $c = 0$, and $d = 0.2$.

- Since $a = 1$, there is neither a vertical stretch nor a vertical compression.
- Since $b = 3/2$ (hence $b > 1$), there is a horizontal compression.
- Since $c = 0$, there is no horizontal shift.
- Since $d = 0.2$ (hence $d \neq 0$), there is a vertical shift.

Now we will show that none of the other answer choices is correct.

The function given in answer choice A, $\cos\theta - 2.9$, is in the form $a\cos(b(\theta + c)) + d$ with $a = 1$, $b = 1$, $c = 0$, and $d = -2.9$. Since $d = -2.9$, there is a vertical shift, but the fact that $b = 1$ shows that there is no horizontal compression.

The function given in answer choice B, $2\cos(\theta + (\pi/8))$, is in the form $a\cos(b(\theta + c)) + d$ with $a = 2$, $b = 1$, $c = \pi/8$, and $d = 0$. Since $b = 1$, there is no

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horizontal compression, and the fact that $d = 0$ means that there is no vertical shift.

The function given in answer choice D, $\cos(0.25\,\theta - (\pi/3))$, can be expressed as

$$\cos\left(\frac{\theta}{4} - \frac{\pi}{3}\right)$$

We can in turn express

$$\frac{\theta}{4} - \frac{\pi}{3}$$

as follows:

$$\frac{\theta}{4} - \frac{\pi}{3} = \left(\frac{1}{4}\right)\left[4\left(\frac{\theta}{4}\right) - 4\left(\frac{\pi}{3}\right)\right] = \left(\frac{1}{4}\right)\left(\theta - \frac{4\pi}{3}\right)$$

Thus

$$\cos\left(\frac{\theta}{4} - \frac{\pi}{3}\right) = \cos\left(\left(\frac{1}{4}\right)\left(\theta - \frac{4\pi}{3}\right)\right)$$

This is in the form $a\cos(b(\theta + c)) + d$ with $a = 1$, $b = 1/4$, $c = -4\pi/3$, and $d = 0$. Since $b = 1/4$ (hence $b < 1$), there is a horizontal stretch (not a horizontal compression), and the fact that $d = 0$ means that there is no vertical shift.

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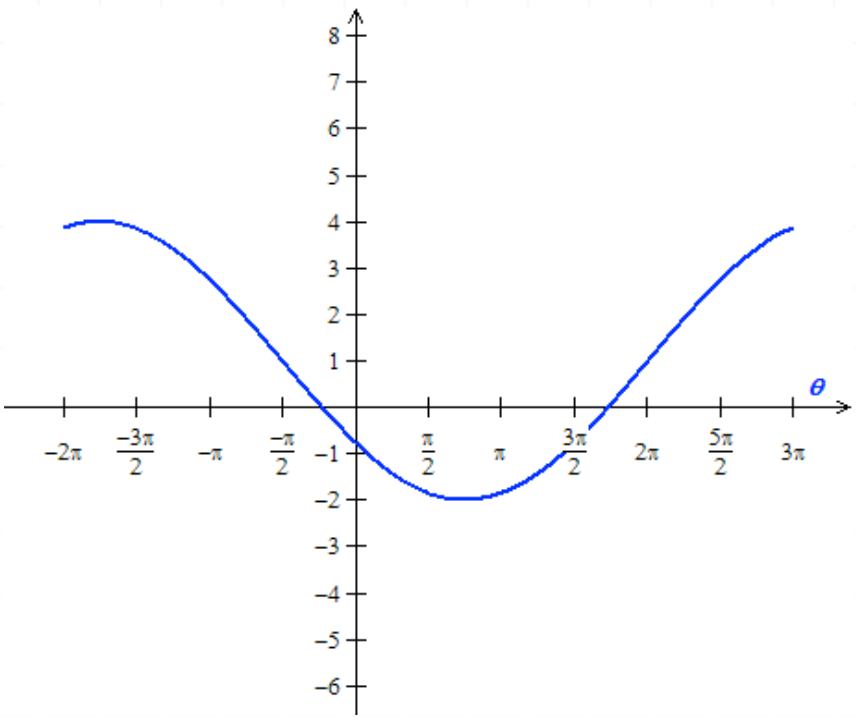
Question: The following curve is the graph of some sine or cosine function over the interval $[-2\pi, 3\pi)$, and its period is 5π. What is the function?



Answer choices:

A $3\cos\left(\frac{\theta}{5}+\frac{\pi}{10}\right)-1$

B $-3\sin\left(\frac{2\theta}{5}+\frac{\pi}{5}\right)+1$

C $2\sin\left(\frac{2\theta}{5}+\frac{\pi}{10}\right)+2$

D $-2\cos\left(\frac{\theta}{10}+\frac{2\pi}{5}\right)-1$

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Solution: B

The values of the function whose graph is shown in the curve range from -2 to 4, so the amplitude of the function is

$$\left(\frac{1}{2}\right)[4-(-2)] = \left(\frac{1}{2}\right)(6) = 3$$

Therefore, we can rule out answer choices C and D, because the functions given in those answer choices have an amplitude of 2.

Also, note that $4 = 3 + 1$ and $-2 = -3 + 1$. This means that the curve has a vertical shift of 1. This rules out choice A, because the function given in that answer choice has a vertical shift of -1.

To verify that the function given in answer choice B,

$$-3\sin\left(\frac{2\theta}{5}+\frac{\pi}{5}\right)+1$$

has a period of 5π, note that

$$\frac{2\theta}{5}+\frac{\pi}{5} = \left(\frac{2}{5}\right)\left[\left(\frac{5}{2}\right)\left(\frac{2\theta}{5}\right)+\left(\frac{5}{2}\right)\left(\frac{\pi}{5}\right)\right] = \left(\frac{2}{5}\right)\left(\theta+\frac{\pi}{2}\right)$$

Therefore, the function given in answer choice B is

$$-3\sin\left(\left(\frac{2}{5}\right)\left(\theta+\frac{\pi}{2}\right)\right)+1$$

This indicates that the period of this function is

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$$\frac{2\pi}{\left(\frac{2}{5}\right)} = 2\pi\left(\frac{5}{2}\right) = 5\pi$$

As a final piece of evidence that the given curve is the graph of the function given in answer choice B, let's determine the values of θ at which that function is equal to 1. Note that those are the values of θ for which

$$\sin\left(\frac{2\theta}{5} + \frac{\pi}{5}\right) = 0$$

Recall that the sine of an angle is equal to 0 if and only if that angle is equal to $n\pi$ for some integer n. Therefore,

$$\frac{2\theta}{5} + \frac{\pi}{5} = n\pi$$

Solving this equation for θ:

$$\frac{2\theta}{5} = n\pi - \frac{\pi}{5}$$

$$\frac{2\theta}{5} = \frac{5(n\pi) - 1(\pi)}{5}$$

$$\frac{2\theta}{5} = \frac{(5n-1)\pi}{5}$$

$$\theta = \left(\frac{5}{2}\right)\left[\frac{(5n-1)\pi}{5}\right]$$

$$\theta = \frac{(5n-1)\pi}{2}$$

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Let's determine the integers n for which $(5n-1)\pi/2$ lies in the interval $[-2\pi,3\pi)$:

$$-2\pi \leq \frac{(5n-1)\pi}{2} \leq 3\pi$$

$$-4\pi \leq (5n-1)\pi \leq 6\pi$$

$$-4\pi \leq 5n\pi - \pi \leq 6\pi$$

$$-4\pi + \pi \leq 5n\pi \leq 6\pi + \pi$$

$$-3\pi \leq 5n\pi \leq 7\pi$$

$$-3 \leq 5n \leq 7$$

$$-\frac{3}{5} \leq n \leq \frac{7}{5}$$

The only integers n that satisfy this pair of inequalities are 0 and 1. Setting n to 0 and 1 in the expression we found for θ, namely $(5n-1)\pi/2$, gives $\theta = -\pi/2$ and $\theta = 4\pi/2 (= 2\pi)$, respectively. Inspection of the given curve shows that the function whose graph it represents is indeed equal to 1 at $\theta = -\pi/2$ and $\theta = 2\pi$.

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